

# Orbital Functional Geometry XVII

The Fundamental Equation in Higher Dimensions:  
Eikonal Structure, Orbital Geodesics, and Tensor Generalisation

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## Abstract

We generalise the fundamental equation  $e^{P(x)(az+b)} = f(x)$  of *Orbital Functional Geometry I* (2026) to higher dimensions.

The natural generalisation with geometric content arises when  $z = \nabla\phi$  is constrained to be a gradient field. The equation  $e^{P(x)(a|\nabla\phi|+b)} = f(x)$  then becomes the *orbital eikonal equation*:

$$|\nabla\phi(x)|^2 = \frac{Q(x)^2}{a^2}, \quad Q = \frac{\ln f}{P} - b$$

Its characteristics are the *orbital geodesics*: curves  $\gamma$  satisfying  $\ddot{\gamma} = (Q/a^2)\nabla Q$ . Orbital geodesics are accelerated by the gradient of the orbital invariant  $Q = \ln f/P - b$ .

The associated orbital Lagrangian and Hamiltonian are:

$$L_{\mathcal{O}}(x, \dot{x}) = \frac{1}{2}|\dot{x}|^2 - V_{\mathcal{O}}(x), \quad H_{\mathcal{O}}(x, p) = \frac{1}{2}|p|^2 + V_{\mathcal{O}}(x)$$

where the orbital potential  $V_{\mathcal{O}}(x) = -Q(x)^2/2a^2$  is attractive (negative). Canonical quantisation gives the orbital Schrödinger operator  $\hat{H}_{\mathcal{O}}^{(n)} = -(\hbar^2/2m)\Delta - (\hbar^2/2ma^2)Q^2$ , which admits bound states (discrete negative spectrum) when  $Q \in L^2(\mathbb{R}^n)$ .

The tensor generalisation replaces  $P$  by a  $(0, 2)$ -tensor field  $P_{\mu\nu}$  and  $z$  by a  $(2, 0)$ -tensor  $z^{\mu\nu}$ . When  $P_{\mu\nu} = g_{\mu\nu}$  (the Riemannian metric) and  $z^{\mu\nu} = G^{\mu\nu}$  (the Einstein tensor), the orbital invariant equals the scalar curvature:  $Q/a = R_g$ . The fundamental equation in tensor form connects  $f$  to the geometry of spacetime through the scalar curvature.

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# Introduction

The fundamental equation  $e^{P(x)(az+b)} = f(x)$  was posed in dimension one. Its solution  $z = (1/a)(\ln f/P - b)$  identified  $u = \ln f/P$  as the natural variable and led to the entire structure of  $\mathcal{O}$ .

In higher dimensions, the naive generalisation (replace  $x \in \mathbb{R}$  by  $x \in \mathbb{R}^n$ , keep  $z$  scalar) adds nothing structurally new. The interesting generalisation requires  $z$  to carry geometric information about  $\mathbb{R}^n$  or  $(M, g)$ .

This paper develops three levels of generalisation: scalar  $z$  with gradient constraint (eikonal structure), vector  $z$  with Hamiltonian structure, and tensor  $z$  with connection to Einstein's equations.

# 1 The Orbital Eikonal Equation

**Definition 1** (Orbital eikonal equation). *Let  $\phi : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$  and  $z = \nabla\phi$ . The fundamental equation  $e^{P(x)(a|\nabla\phi|+b)} = f(x)$  becomes, after taking logarithms and squaring:*

$$|\nabla\phi(x)|^2 = \frac{Q(x)^2}{a^2}$$

where  $Q(x) = \ln f(x)/P(x) - b$  is the orbital invariant of Paper I. This is the orbital eikonal equation.

**Remark 1** (Classical eikonal equation). *The classical eikonal equation of geometric optics is  $|\nabla\phi|^2 = n(x)^2$ , where  $n(x)$  is the refractive index. The orbital eikonal equation has the same form with  $n(x) = |Q(x)|/a$  — the orbital invariant plays the role of refractive index. High  $|Q|$  (far from the orbital equilibrium  $f = e^{bP}$ ) corresponds to high refractive index: the orbital “light” slows down near the zeros of  $Q$ .*

**Theorem 1** (Characteristics: orbital geodesics). *The characteristics of the orbital eikonal equation — the curves along which the eikonal propagates — satisfy:*

$$\ddot{\gamma}(t) = \frac{Q(\gamma)}{a^2} \nabla Q(\gamma) = \frac{1}{2a^2} \nabla Q^2(\gamma)$$

These are the orbital geodesics. They are accelerated by the gradient of  $Q^2$  — equivalently, by the gradient of  $(\ln f/P - b)^2$ .

*Proof.* Differentiating  $|\nabla\phi|^2 = Q^2/a^2$  along a characteristic  $\gamma$  with  $\dot{\gamma} = \nabla\phi$ :  $\frac{d}{dt}|\dot{\gamma}|^2 = 2\langle\dot{\gamma}, \ddot{\gamma}\rangle = \frac{2Q}{a^2}\langle\nabla\phi, \nabla Q\rangle = \frac{2Q}{a^2}\dot{\gamma} \cdot \nabla Q$ . Matching components:  $\ddot{\gamma} = (Q/a^2)\nabla Q$ .  $\square$

**Remark 2** (Zeros of  $Q$  as orbital fixed points). *At a zero of  $Q$  (i.e.  $f(x) = e^{bP(x)}$ ),  $\nabla Q^2 = 2Q\nabla Q$ . If  $Q = 0$  and  $\nabla Q \neq 0$ , the acceleration vanishes:  $\ddot{\gamma} = 0$ . Orbital geodesics pass through zeros of  $Q$  with constant velocity — they are not attracted to or repelled from them, but pass through freely. The attractive behaviour occurs near large  $|Q|$ , not at zeros.*

# 2 Orbital Lagrangian and Hamiltonian

**Definition 2** (Orbital Lagrangian). *The Lagrangian for which orbital geodesics are extremals of zero energy:*

$$L_O(x, \dot{x}) = \frac{1}{2}|\dot{x}|^2 - V_O(x), \quad V_O(x) = -\frac{Q(x)^2}{2a^2}$$

Orbital geodesics are zero-energy curves:  $\frac{1}{2}|\dot{\gamma}|^2 = V_O(\gamma)$ , i.e.  $|\dot{\gamma}|^2 = Q^2/a^2$ .

**Remark 3** (Attractive potential).  $V_O(x) = -Q(x)^2/2a^2 \leq 0$  everywhere. The potential is attractive (negative), vanishing only at zeros of  $Q$ . Orbital geodesics are confined to regions where  $V_O \leq 0$ , which is everywhere — the orbital dynamics is globally defined.

**Definition 3** (Orbital Hamiltonian). *By Legendre transform ( $p = \dot{x}$ ):*

$$H_O(x, p) = \frac{1}{2}|p|^2 + V_O(x) = \frac{1}{2}|p|^2 - \frac{Q(x)^2}{2a^2}$$

Hamilton's equations:

$$\dot{x} = p, \quad \dot{p} = -\nabla V_{\mathcal{O}} = \frac{Q}{a^2} \nabla Q$$

The second equation reproduces the orbital geodesic equation:  $\ddot{x} = (Q/a^2) \nabla Q$ .

**Theorem 2** (Energy conservation). *Along orbital geodesics,  $H_{\mathcal{O}} = \frac{1}{2}|p|^2 - Q^2/2a^2 = 0$ : the orbital Hamiltonian is conserved at zero. The zero-energy surface of  $H_{\mathcal{O}}$  is the phase space of orbital geodesics.*

### 3 Canonical Quantisation

**Definition 4** (Orbital Schrödinger operator in  $\mathbb{R}^n$ ). *Replacing  $p \rightarrow -i\hbar\nabla$ :*

$$\hat{H}_{\mathcal{O}}^{(n)} = -\frac{\hbar^2}{2m}\Delta - \frac{\hbar^2}{2ma^2}Q(x)^2$$

*This is a Schrödinger operator with attractive potential  $V_{\mathcal{O}}(x) = -(\hbar^2/2ma^2)Q(x)^2$ .*

**Theorem 3** (Bound states). *If  $Q \in L^2(\mathbb{R}^n)$ , i.e.  $\int_{\mathbb{R}^n} Q(x)^2 dx < \infty$ , then  $\hat{H}_{\mathcal{O}}^{(n)}$  admits at least one bound state (negative eigenvalue) in dimensions  $n \leq 3$  by the Birman–Schwinger principle.*

*The bound state energies  $E_k < 0$  correspond to orbital states localised near regions of large  $|Q|$  — the orbital wells of Paper IV, generalised to dimension  $n$ .*

**Remark 4** (Connection to Paper IV). *In Paper IV, the energy wells of  $\zeta$  were the non-trivial zeros  $s_k = 1/2 + it_k$ , where  $Q \rightarrow -\infty$ . In dimension  $n$ , the wells are the zeros of  $Q(x) = \ln f(x)/P(x) - b$ , i.e. the level set  $f(x) = e^{bP(x)}$ . The structure of wells is dimension-independent: it depends on  $f$  and  $P$ , not on  $n$ .*

### 4 Tensor Generalisation

**Definition 5** (Fundamental equation in tensor form). *Let  $P_{\mu\nu}$  be a  $(0,2)$ -tensor field and  $z^{\mu\nu}$  a  $(2,0)$ -tensor on  $(M, g)$ . The tensor fundamental equation is:*

$$e^{P_{\mu\nu}(a z^{\mu\nu} + b g^{\mu\nu})} = f$$

*where the exponent uses the contraction  $P_{\mu\nu}z^{\mu\nu} = \text{tr}(P \cdot z)$ .*

**Theorem 4** (Tensor orbital invariant). *The solution for the trace  $\text{tr}(z) = g_{\mu\nu}z^{\mu\nu}$ :*

$$\text{tr}(z) = \frac{1}{a} \left( \frac{\ln f}{P_{\mu\nu}g^{\mu\nu}} - b \right) = \frac{Q_g}{a}$$

*where  $Q_g = \ln f / \text{tr}_g(P) - b$  is the tensor orbital invariant.*

**Theorem 5** (Einstein connection). *Set  $P_{\mu\nu} = g_{\mu\nu}$  (the Riemannian metric) and  $z^{\mu\nu} = G^{\mu\nu}$  (the Einstein tensor  $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}$ ). Then:*

$$P_{\mu\nu}z^{\mu\nu} = g_{\mu\nu}G^{\mu\nu} = R - \frac{n}{2}R = \left(1 - \frac{n}{2}\right) R$$

where  $R$  is the scalar curvature. The tensor fundamental equation becomes:

$$e^{a(1-n/2)R+nb} = f$$

giving:

$$R = \frac{\ln f - nb}{a(1 - n/2)} = \frac{Q_g}{a(1 - n/2)}$$

The scalar curvature of  $(M, g)$  equals the tensor orbital invariant (up to a dimensional factor).

**Corollary 1** (In dimension  $n = 4$ ). For  $n = 4$  (spacetime):  $1 - n/2 = -1$ , giving  $R = -Q_g/a$ . The fundamental equation  $e^{-aR+4b} = f$  relates  $f$  to the scalar curvature of spacetime. If  $f = e^{4b}$  (constant), then  $R = 0$ : Ricci-flat spacetime is the “orbital equilibrium” of the tensor equation in dimension 4.

**Remark 5** (Orbital equilibrium and Ricci-flatness). The orbital equilibrium  $Q = 0$  (i.e.  $f = e^{bP}$ ) generalises to  $R = 0$  in the tensor case. Ricci-flat manifolds (Calabi–Yau, Schwarzschild vacuum) are the orbital equilibria of the tensor fundamental equation. This gives a new characterisation of Ricci-flatness: it is the zero of the tensor orbital invariant.

## 5 The Hierarchy of Fundamental Equations

| Level      | Equation                                        | $z$                           | Invariant                  |
|------------|-------------------------------------------------|-------------------------------|----------------------------|
| Scalar (I) | $e^{P(az+b)} = f$                               | $z \in \mathbb{R}$            | $Q = \ln f/P - b$          |
| Eikonal    | $ \nabla\phi ^2 = Q^2/a^2$                      | $\nabla\phi \in \mathbb{R}^n$ | $Q$ as refractive index    |
| Vector     | $H_{\mathcal{O}} = \frac{1}{2} p ^2 - Q^2/2a^2$ | $p \in T^*M$                  | $Q$ as potential           |
| Tensor     | $e^{P_{\mu\nu}(az^{\mu\nu}+bg^{\mu\nu})} = f$   | $z^{\mu\nu} \in T^{(2,0)}M$   | $Q_g = R \cdot a(1 - n/2)$ |

At each level,  $Q = \ln f/P - b$  remains the central invariant. The dimension increases but the orbital invariant is unchanged in structure.

# Summary of New Results

| Result                        | Statement                                                              | Status      |
|-------------------------------|------------------------------------------------------------------------|-------------|
| Orbital eikonal               | $ \nabla\phi ^2 = Q^2/a^2$                                             | Established |
| $Q$ as refractive index       | $ Q /a$ plays role of $n(x)$                                           | Established |
| Orbital geodesics             | $\ddot{\gamma} = (Q/a^2)\nabla Q$                                      | Established |
| Zeros of $Q$ as fixed points  | Geodesics pass freely through $Q = 0$                                  | Established |
| Orbital Lagrangian            | $L_{\mathcal{O}} = \frac{1}{2} \dot{x} ^2 + Q^2/2a^2$                  | Established |
| Orbital Hamiltonian           | $H_{\mathcal{O}} = \frac{1}{2} p ^2 - Q^2/2a^2$                        | Established |
| Zero energy surface           | Orbital geodesics on $H_{\mathcal{O}} = 0$                             | Established |
| Bound states                  | Birman–Schwinger for $Q \in L^2$ , $n \leq 3$                          | Established |
| Wells in dim $n$              | Level set $f = e^{bP}$ independent of $n$                              | Established |
| Tensor equation               | $e^{P_{\mu\nu}(az^{\mu\nu}+bg^{\mu\nu})} = f$                          | Established |
| Einstein connection           | $R = Q_g/a(1 - n/2)$ for $z^{\mu\nu} = G^{\mu\nu}$                     | Established |
| Ricci-flat = equilibrium      | $R = 0 \iff Q_g = 0$ in dim 4                                          | Established |
| Hierarchy table               | Scalar $\rightarrow$ eikonal $\rightarrow$ vector $\rightarrow$ tensor | Established |
| Orbital geodesics on $(M, g)$ | Curved space eikonal                                                   | Open        |
| Tensor $Z_{\mathcal{O}}$      | Zeta function of tensor orbital invariant                              | Open        |

## Acknowledgements

This paper returns to the fundamental equation of *Orbital Functional Geometry I* (2026) and asks what it means in higher dimensions. The eikonal structure emerged from constraining  $z$  to be a gradient field — the simplest geometric constraint in  $\mathbb{R}^n$ . The tensor generalisation was forced by asking what  $P_{\mu\nu}z^{\mu\nu}$  gives when  $z^{\mu\nu} = G^{\mu\nu}$ : the scalar curvature appeared without being sought. The identification of Ricci-flat manifolds as orbital equilibria was not anticipated.

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## References

- [1] J. Brindel, *Orbital Functional Geometry I–XVI*, Preprints, Cotonou, 2026.
- [2] L. C. Evans, *Partial Differential Equations*, AMS, 2nd ed., 2010.
- [3] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*, Freeman, 1973.
- [4] M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV: Analysis of Operators*, Academic Press, 1978.
- [5] M. P. do Carmo, *Riemannian Geometry*, Birkhäuser, 1992.
- [6] I. Chavel, *Eigenvalues in Riemannian Geometry*, Academic Press, 1984.